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Application of local linking to asymptotically linear wave equations with resonance II

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Abstract. Existence of a time-periodic solution to a non-linear wave equation with resonance is established by a variational method. We consider the 2π -periodic weak solution to a wave equation $\square u(x, t) = h(x, t, u(x, t))$ of space dimension 1, where $h(x, t, \xi)$ is asymptotically linear in ξ both as $\xi \rightarrow 0$ or $\xi \rightarrow \infty$, with the co-efficient as $\xi \rightarrow \infty$ belonging to $\sigma(\square)$. It was proved that there are some cases, where the difference of $h(t, x, \xi)$ from its linear approximation is not bounded, that guarantee the existence of a non-trivial weak solution. In this paper, we show that the restriction in our previous result imposed on these co-efficients can be further relaxed.

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§1. Introduction

The purpose of this paper is to extend the result of [12], which is concerned with the existence of a non-trivial time-periodic solution to the following non-linear wave equation (WE) with asymptotically linear non-linear term h ($\square := \partial^2/\partial t^2 - \partial^2/\partial x^2$):

$$(WE) \quad \begin{cases} \square u(x, t) = h(x, t, u(x, t)), & (0 < x < \pi, \ t \in \mathbb{R}), \\ u(0, t) = u(\pi, t) = 0, & (t \in \mathbb{R}), \\ u(x, t + 2\pi) = u(x, t), & (0 < x < \pi, \ t \in \mathbb{R}). \end{cases}$$

Many authors treated this problem by variational methods under various conditions on $h(x, t, \xi)$ ([1]-[5], [7]-[9], [11], [12], [14], [16]-[18]). In this paper we treat the case where the non-linear term h is asymptotically linear at both

0 and ∞ in the following sense: There exist constants b_0 and b for which

$$\begin{aligned} g_0(x, t, \xi) &:= h(x, t, \xi) - b_0\xi = o(|\xi|) \text{ as } \xi \rightarrow 0 \text{ uniformly in } (x, t), \\ g(x, t, \xi) &:= h(x, t, \xi) - b\xi = o(|\xi|) \text{ as } |\xi| \rightarrow \infty \text{ uniformly in } (x, t). \end{aligned}$$

Li and Szulkin [8], Kryszewski and Szulkin [7] and Bartsch and Ding [2] already considered the case where $h(x, t, \xi)$ is asymptotically linear in ξ both as $\xi \rightarrow 0$ and $|\xi| \rightarrow \infty$. However, they all assume that $h(x, t, \xi) - b\xi$ is bounded when $b \in \sigma(\square)$ (“resonant” case). Miyajima and Tanaka [12] treated the case where $h(x, t, \xi) - b\xi$ is not bounded and $b, b_0 \in \sigma(\square)$. Their results shows, for example, that (WE) has a non-trivial periodic solution for $h(x, t, \xi) = b\xi + |\xi|^\alpha \operatorname{sgn} \xi$ with $0 < \alpha < 1$. (To be rigorous, $|\xi|^\alpha \operatorname{sgn} \xi$ should be deformed to a C^2 class function in a neighborhood of 0.)

The main purpose of this paper is to show that the condition imposed on b_0 and b can be further relaxed under the same assumption (C2) in [12] (see Section 3) on the nonlinear term (see Remark 21). Another purpose is to prove the existence of a non-trivial solution to (WE) when $h(x, t, \xi)$ is odd in ξ , by using Krasnoselskii genus. Note that Bartsch and Ding [2] considered the case of even functional corresponding to (WE) using the equivariant limit category.

In the following Section 2, we firstly obtain an abstract existence theory of a non-trivial critical point for C^1 -class functional, its proof is based on that of Bartsch and Ding [2]. We also have Lusternik-Schnirelmann-type results for even functional. In Section 3, we show the existence of a non-trivial weak solution to (WE).

§2. Some results on critical point theory

Throughout this section, E denotes a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and Φ denotes a C^1 class functional on E . The gradient $\nabla\Phi(u)$ ($u \in E$) is considered to be an element of E through the Riesz representation theorem. A subset \tilde{E} is defined by $\tilde{E} := \{u \in E : \nabla\Phi(u) \neq 0\}$. Then recall that a map $V: \tilde{E} \rightarrow E$ is called a pseudo-gradient vector field (abbreviated to “p.g.v.f.”) for Φ if V satisfies the following conditions for every $u \in \tilde{E}$:

$$\begin{cases} \|V(u)\| \leq \frac{3}{2} \|\nabla\Phi(u)\|, \\ \langle \nabla\Phi(u), V(u) \rangle \geq \frac{1}{2} \|\nabla\Phi(u)\|^2. \end{cases}$$

It is well known that there exists a locally Lipschitz continuous pseudo-gradient vector field V for every C^1 class functional Φ ([13, Lemma 6.1]). For such V , the ordinary differential equation

$$(2.1) \quad \frac{du(t)}{dt} = V(u(t)), \quad u(0) = u_0 \quad (u_0 \in \tilde{E})$$

has a unique solution which is maximally defined in the positive direction of t .

2.1. An invariant set with respect to gradient flow and minimax method

Definition 1 *Let U be a subset of E . We say that U is an invariant set with respect to p.g.v.f. V for Φ if $\eta(t, u) \in U$ holds for every $u \in U \cap \tilde{E}$ and $t \geq 0$, where $\eta(t, u)$ is the maximal solution of differential equation (2.1) starting from u .*

Now we consider a condition for a subset U to be invariant with respect to some p.g.v.f. V for Φ . The following proposition was shown in [12].

Proposition 2 ([12, Proposition 1]) *Let $f: E \rightarrow \mathbb{R}$ be a C^1 class functional on E and let $U := \{x \in E \mid f(x) < 0\}$ be non-empty. Suppose that Φ satisfies*

$$(2.2) \quad \langle \nabla \Phi, \nabla f(u) \rangle > 0 \quad \text{on } \partial U.$$

Then there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ such that U is invariant with respect to $-V$.

Next we prepare the following lemma, which is a variation of minimax argument using some family of invariant set. (see [15, Theorem 4.2.], [19, Theorem 2.8.])

Lemma 3 *Let M be a compact subset of E and M_0 a closed subset of M . Moreover suppose that U is an invariant subset of E with respect to some p.g.v.f. V for Φ . Then we define;*

$$\Sigma := \{ \gamma \in C([0, 1] \times M, E) \mid \gamma \text{ satisfies the following condition (A)} \}$$

$$\begin{aligned} c &:= \sup_{\gamma \in \Sigma} \min_{(t, u) \in [0, 1] \times M} \Phi(\gamma(t, u)), \quad c_0 := \min_{u \in M} \Phi(u), \\ (A) \quad &\left\{ \begin{array}{l} \text{(a) } \gamma(0, u) = u \text{ for } u \in M \text{ and } \gamma(1, M) \subset U \\ \text{(b) there exists some constant } t_\gamma \in (0, 1) \text{ such that} \\ \quad \Phi(\gamma(t, u)) \geq \Phi(u) \text{ for } (t, u) \in [0, t_\gamma] \times M_0 \\ \quad \text{and } \gamma(t, u) \in U \text{ for } (t, u) \in [t_\gamma, 1] \times M_0. \end{array} \right. \end{aligned}$$

If $\Sigma \neq \emptyset$ and $c_0 > c$ hold, then for every $\varepsilon > 0$ there exists a $v \in E$ such that

$$c - 4\varepsilon \leq \Phi(v) \leq c + 4\varepsilon, \quad \|\nabla \Phi(v)\| \leq 4\varepsilon.$$

Proof. Without loss of generality, we can assume that $\varepsilon > 0$ satisfies $c_0 > c + 4\varepsilon$. Suppose that

$$\|\nabla\Phi(v)\| > 4\varepsilon \quad \text{for every } v \in \Phi^{-1}([c - 4\varepsilon, c + 4\varepsilon]).$$

By the definition of c , there exists some $\gamma_0 \in \Sigma$ such that

$$(2.3) \quad \min_{(t,u) \in [0,1] \times M} \Phi(\gamma_0(t, u)) \geq c - \varepsilon.$$

Let $f : E \rightarrow [0, 1]$ be a locally Lipschitz continuous function such that

$$f(u) := \begin{cases} 1 & \text{if } u \in \Phi^{-1}([c - 2\varepsilon, c + 2\varepsilon]), \\ 0 & \text{if } u \notin \Phi^{-1}([c - 3\varepsilon, c + 3\varepsilon]). \end{cases}$$

We consider the differential equation with the initial value $u \in E$

$$\begin{cases} \frac{d}{ds}\sigma(s, u) = f(\sigma(s, u)) \frac{V(\sigma(s, u))}{\|V(\sigma(s, u))\|}, \\ \sigma(0, u) = u, \end{cases}$$

and let $\sigma(s, u)$ be the maximal solution of the above differential equation. It is verified that $\sigma(s, u)$ is well defined for every $(s, u) \in [0, \infty) \times E$. We define $g(t, u) := \sigma(1, \gamma_0(t, u))$ for $(t, u) \in [0, 1] \times M$.

Then we can show that g satisfies (A). Indeed, for every $u \in M$, we have $\Phi(\gamma_0(0, u)) = \Phi(u) \geq c_0 > c + 4\varepsilon$. Therefore we obtain $g(0, u) = \sigma(1, \gamma_0(0, u)) = \sigma(1, u) = u$ for every $u \in M$. For every $u \in M$, we have $g(1, u) = \sigma(1, \gamma_0(1, u)) \in U$ because U is an invariant subset with respect to V . Moreover since $\gamma_0(t, u) \in U$ for every $(t, u) \in [t_{\gamma_0}, 1] \times M_0$, we obtain $g(t, u) = \sigma(1, \gamma_0(t, u)) \in U$ for every $(t, u) \in [t_{\gamma_0}, 1] \times M_0$. Finally because $\Phi(\sigma(s, u))$ is non-decreasing in s , we obtain $\Phi(g(t, u)) \geq \Phi(\gamma_0(t, u)) \geq \Phi(u)$ for every $(t, u) \in [0, t_{\gamma_0}] \times M_0$. Hence g satisfies (A).

On the other hand if $c - \varepsilon \leq \Phi(\sigma(s, \gamma_0(t, u))) \leq c + 2\varepsilon$ for every $s \in [0, 1]$, then we obtain

$$\begin{aligned} \Phi(g(t, u)) &= \Phi(\sigma(1, \gamma_0(t, u))) \\ &\geq \Phi(\gamma_0(t, u)) + \int_0^1 \frac{1}{3} \|\nabla\Phi(\sigma(s, \gamma_0(t, u)))\| ds \\ &\geq c - \varepsilon + 4\varepsilon/3 = c + \varepsilon/3 \end{aligned}$$

from $\Phi(\gamma_0(t, u)) \geq c - \varepsilon$, which holds by (2.3) and the definition of V . Moreover, if there exists some $s \in [0, 1]$ such that $\Phi(\sigma(s, \gamma_0(t, u))) \geq c + 2\varepsilon$, then we have $\Phi(g(t, u)) \geq \Phi(\sigma(s, \gamma_0(t, u))) \geq c + 2\varepsilon$. Therefore we obtain $\Phi(g(t, u)) \geq c + \frac{\varepsilon}{3}$ for every $(t, u) \in [0, 1] \times M$. This is a contradiction to the definition of c . ■

For later use, we prepare the following slight generalization of Proposition 2 for a U with specific piecewise smooth boundary.

Proposition 4 *Suppose that there exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$, and Φ satisfies the following condition (R). Then there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ on \tilde{E} satisfying condition (R) with $\nabla\Phi$ replaced by V , for which the region*

$$(2.4) \quad U := \{ (v_\infty, w_\infty) \mid \|v_\infty\| > \max\{R_1, \delta\|w_\infty\|^\lambda\} \}$$

is invariant with respect to V .

(R) *The following (i) and (ii) hold for some $\lambda \geq 0$, $\delta > 0$, and $R_1 > 0$, where*

$$u = w_\infty + v_\infty \quad (w_\infty \in W_\infty, v_\infty \in V_\infty).$$

$$(i) \quad \left\langle \nabla\Phi(u), v_\infty - \lambda\delta^2 \frac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0 \quad (\text{if } \|v_\infty\| = \delta\|w_\infty\|^\lambda, \|v_\infty\| \geq R_1).$$

$$(ii) \quad \langle \nabla\Phi(u), v_\infty \rangle > 0 \quad (\text{if } \|v_\infty\| \geq \delta\|w_\infty\|^\lambda, \|v_\infty\| = R_1).$$

We omit the proof because we can prove Proposition 4 by the same argument as that in the proof of Proposition 4 of [12].

Remark 5 In [12], it was proved that there exists a p.g.v.f. V for Φ satisfying condition (R) with $\nabla\Phi$ replaced by V , and the region

$$(2.5) \quad \tilde{U} := \{ (v_\infty, w_\infty) \mid \|v_\infty\| < \max\{R_1, \delta\|w_\infty\|^\lambda\} \}$$

is invariant with respect to $-V$.

Note that \tilde{U} is the complement of closure of U given by (2.4), and condition (R) says that the inner product of $\nabla\Phi$ and outward normal vector to U is negative on ∂U .

2.2. Local linking and the main result

Let us recall the definition of local linking.

Definition 6 *Let Φ be a C^1 functional on E . Then we define as follows.*

- (i) If there exist an orthogonal decomposition $E = V_0 \oplus W_0$ and an $r > 0$ satisfying the following condition, Φ is said to have a local linking at 0 with respect to (V_0, W_0) :

$$(2.6) \quad \begin{cases} \Phi(u) \geq 0 & (\forall u \in B_r V_0), \\ \Phi(u) \leq 0 & (\forall u \in B_r W_0), \end{cases}$$

where, $B_r V_0 := \{u \in V_0 : \|u\| \leq r\}$, $B_r W_0 := \{u \in W_0 : \|u\| \leq r\}$.

- (ii) Φ is said to have a strong local linking at 0 with respect to (V_0, W_0) if there exist an $r > 0$ satisfying (2.6), and the following properties hold for some $\varepsilon > 0$:

$$(2.7) \quad \begin{cases} \Phi(u) \geq \varepsilon & \text{on } \partial B_r V_0, \\ \Phi(u) \leq -\varepsilon & \text{on } \partial B_r W_0. \end{cases}$$

We state the $(WPS)_c^*$ condition introduced by [2], which is a generalization of the usual Palais–Smale condition.

Definition 7 Suppose that a sequence $\{E_n\}_n$ of finite dimensional subspaces of E satisfy

$$(2.8) \quad E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset E, \quad E = \overline{\bigcup_{n=1}^{\infty} E_n},$$

and let P_n denote the orthogonal projection from E onto E_n . Then,

- (i) a sequence $\{u_j\}_j$ is called a $(PS)_c^*$ sequence (with respect to Φ and $\{E_n\}_n$) provided $u_j \in E_{n_j}$, $n_j \rightarrow \infty$, $\Phi(u_j) \rightarrow c$ and $P_{n_j}(\nabla \Phi(u_j)) \rightarrow 0$ (as $j \rightarrow \infty$);
- (ii) Φ is said to satisfy the $(WPS)_c^*$ condition if every $(PS)_c^*$ sequence has a subsequence weakly convergent to a critical point u of Φ with $\Phi(u) = c$.

The following easy paraphrase is useful in proving the existence of a non-trivial critical point.

Lemma 8 Let $\{E_n\}_n$ be as in Definition 7 and let Φ satisfy the $(WPS)_c^*$ condition for every $c \in \mathbb{R}$. Moreover, suppose that 0 is the only critical value of Φ . Then for any $\varepsilon > 0$ and $M > 0$, there exist some $b > 0$ and $n_0 \in \mathbb{N}$ such that

$$(2.9) \quad \|\nabla \Phi_n(u)\| \geq b, \quad \forall u \in \Phi_n^{-1}([-M, -\varepsilon]) \cup \Phi_n^{-1}([\varepsilon, M])$$

holds for every $n \geq n_0$, where $\Phi_n := \Phi|_{E_n}$. (Note that $\nabla \Phi_n(u) = P_n(\nabla \Phi(u))$ for $u \in E_n$.)

Now let us collect the conditions relevant to our main result about the critical points.

- (Φ1) With respect to a sequence $\{E_n\}_n$ of finite dimensional subspaces satisfying (2.8), Φ satisfies $(WPS)_c^*$ condition for every $c \in \mathbb{R}$.
- (Φ2) Φ is bounded on every bounded set.
- (Φ3) There exists an orthogonal decomposition $E = V_0 \oplus W_0$ that satisfies one of the following conditions:
 - (i) Φ has a strong local linking at 0 with respect to (V_0, W_0) .
 - (ii) Φ has a local linking at 0 with respect to (V_0, W_0) , and for some $r > 0$ with the property (2.6), every $(PS)_0^*$ sequence in $B_{2r}E$ has a strongly convergent subsequence.
- (Φ4) There exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$ that satisfies the following (i)~(iii) for some $\lambda \geq 0$, $\delta > 0$, $R_1 > 0$:

- (i) $\left\langle \nabla \Phi(u), v_\infty - \lambda \delta^2 \frac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0$, (if $\|v_\infty\| = \delta \|w_\infty\|^\lambda$, $\|v_\infty\| \geq R_1$),
- (ii) $\langle \nabla \Phi(u), v_\infty \rangle > 0$, (if $\|v_\infty\| \geq \delta \|w_\infty\|^\lambda$, $\|v_\infty\| \geq R_1$),
- (iii) for every $c < 0$ there exists an $R > 0$ such that $\Phi(u) < c$ provided $\|v_\infty\| \leq \delta \|w_\infty\|^\lambda$ and $\|w_\infty\| \geq R$,

where

$$u = w_\infty + v_\infty \quad (w_\infty \in W_\infty, v_\infty \in V_\infty).$$

Remark 9 We denote that if Φ has a local linking at 0 with respect to (V_0, W_0) , then 0 is a critical point of Φ . Therefore if the assumption (Φ3) holds, then 0 is a critical point of Φ .

When we assume the conditions (Φ1) and (Φ3), we adopt the following notations:

$$\begin{aligned} \Phi_n &:= \Phi|_{E_n}, \\ \Phi_n^c &:= \{u \in E_n : \Phi(u) \leq c\}, & (\Phi_n)_c &:= \{u \in E_n : \Phi(u) \geq c\}, \\ V_0^n &:= E_n \cap V_0, & W_0^n &:= E_n \cap W_0, \\ V_\infty^n &:= E_n \cap V_\infty, & W_\infty^n &:= E_n \cap W_\infty. \end{aligned}$$

We say that the sequence $\{E_n\}_n$ in (Φ1) is *compatible* with the orthogonal decomposition $V_0 \oplus W_0$ [resp. $V_\infty \oplus W_\infty$] (Φ1) if

$$(2.10) \quad E_n = (E_n \cap V_0) \oplus (E_n \cap W_0) \quad [\text{resp. } E_n = (E_n \cap V_\infty) \oplus (E_n \cap W_\infty)]$$

holds for every n . The next lemma is found as Lemma 6.5 in [13] and it can be proved by the standard deformation argument. (cf. [19, Lemma 2.3])

Lemma 10 (Deformation Lemma) *Suppose $(\Phi 1)$ and (ii) of $(\Phi 3)$ hold and there exists no non-zero critical point of Φ in $B_{2r}E$, where $r > 0$ satisfies (2.6) in $(\Phi 3)$. Then there exist some $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist continuous deformations $\xi_n, \eta_n \in C([0, 1] \times E_n, E_n)$ satisfying the following conditions:*

- (i) $\xi_n(0, \cdot) = \eta_n(0, \cdot) = id$,
- (ii) $\xi_n(t, \cdot), \eta_n(t, \cdot)$ are homeomorphisms from E_n to E_n for every $t \in [0, 1]$
- (iii) $\|\xi_n(t, u) - u\| \leq \frac{r}{2}, \|\eta_n(t, u) - u\| \leq \frac{r}{2}$ for every $(t, u) \in [0, 1] \times E_n$,
- (iv) $\sup \Phi \circ \xi_n([0, 1] \times B_r W_0^n) = \inf \Phi \circ \eta_n([0, 1] \times B_r V_0^n) = 0$,
- (v) $\Phi \circ \xi_n(t, \cdot) |_{\partial B_r W_0^n} < 0, \Phi \circ \eta_n(t, \cdot) |_{\partial B_r V_0^n} > 0$ for every $t \in (0, 1]$,
- (vi) $\xi_n(1, u) \subset \Phi_n^{-\varepsilon}$ for every $u \in (B_{\frac{3}{2}r}E \cap \Phi_n^\varepsilon) \setminus B_{\frac{r}{3}}E$,
- (vii) $\eta_n(1, u) \subset (\Phi_n)_\varepsilon$ for every $u \in (B_{\frac{3}{2}r}E \cap (\Phi_n)_{-\varepsilon}) \setminus B_{\frac{r}{3}}E$.

Now we prepare the following definition and lemma to prove our main result.

Definition 11 *Suppose that $(\Phi 3)$ holds, then we define as follows for $n \in \mathbb{N}$. Note that condition (A2) makes sense for sufficiently large n , since there appears the mapping η_n in Lemma 10.*

- (i) *In the case where (i) of $(\Phi 3)$ holds, a mapping $\gamma \in C([0, 1] \times B_r V_0^n, E_n)$ is said to satisfy the condition (A1) for a subset U if the following conditions (a) and (b) hold.*
 - (a) $\gamma(0, u) = u$ for every $u \in B_r V_0^n$ and $\gamma(1, B_r V_0^n) \subset U$.
 - (b) *There exists some $t_\gamma > 0$ such that $\Phi(\gamma(t, u)) \geq \Phi(u)$ for every $(t, u) \in [0, t_\gamma] \times \partial B_r V_0^n$ and $\gamma(t, u) \in U$ for every $(t, u) \in [t_\gamma, 1] \times \partial B_r V_0^n$*
- (ii) *Assume that 0 is the only critical point of Φ in $B_{2r}E$. In the case where (ii) of $(\Phi 3)$ holds, a mapping $\gamma \in C([0, 1] \times B_r V_0^n, E_n)$ is said to satisfy the condition (A2) for a subset U if the following conditions (a) and (b) hold, where η_n denotes a mapping satisfying (i) \sim (vii) in Lemma 10 and $r > 0$ denotes a constant satisfying (2.6).*
 - (a) $\gamma(0, u) = u$ for every $u \in B_r V_0^n$ and $\gamma(1, B_r V_0^n) \subset U$.

- (b) *There exist some $t_\gamma^1 \in (0, 1)$ and $t_\gamma^2 \in [t_\gamma^1, 1]$ such that $\gamma(t, u) = \eta_n(t/t_\gamma^1, u)$ for every $(t, u) \in [0, t_\gamma^1] \times \partial B_r V_0^n$, $\Phi(\gamma(t, u)) \geq \Phi(\eta_n(1, u))$ for every $(t, u) \in [t_\gamma^1, t_\gamma^2] \times \partial B_r V_0^n$ and $\gamma(t, u) \in U$ for every $(t, u) \in [t_\gamma^2, 1] \times \partial B_r V_0^n$*

Now we state a linking lemma where condition (A1) or (A2) above is concerned. The proof is based on a standard argument using degree theory (cf. [2, Lemma 3.2.], [10])

Lemma 12 *Suppose that $(\Phi 3)$ holds and $\{E_n\}$ is compatible with respect to $(V_0 \oplus W_0)$. Let U be a subset of E satisfying $\text{dist}(0, U) \geq 2r$, where $r > 0$ satisfies (2.6). Then the following (i) and (ii) hold:*

- (i) *If the case (i) of $(\Phi 3)$ holds, then $\gamma \in C([0, 1] \times B_r V_0^n, E_n)$ satisfying (A1) for U has the property*

$$\gamma([0, 1] \times B_r V_0^n) \cap \partial B_r W_0^n \neq \emptyset$$

for every $n \in \mathbb{N}$.

- (ii) *Assume that 0 is the only critical point in $B_{2r}E$. If the case (ii) of $(\Phi 3)$ holds, then $\gamma \in C([0, 1] \times B_r V_0^n, E_n)$ satisfying (A2) for U has the property*

$$\gamma([0, 1] \times B_r V_0^n) \cap \xi_n(1, \partial B_r W_0^n) \neq \emptyset$$

for every $n \geq n_0$, where ξ_n denotes a mapping satisfying (i) \sim (vi) in Lemma 10 and $n_0 \in \mathbb{N}$ denotes a natural number in Lemma 10.

Proof. First we prove the case (i) by contradiction. So we suppose that $\gamma \in C([0, 1] \times B_r V_0^n, E_n)$ satisfies (A1) for U and

$$(2.11) \quad \gamma([0, 1] \times B_r V_0^n) \cap \partial B_r W_0^n = \emptyset.$$

Set $\Omega_n := \text{int}(B_r V_0^n \times B_r W_0^n)$. Since γ satisfies (A1) for U , we have for every $(u_1, u_2) \in \partial B_r V_0^n \times B_r W_0^n$

$$\Phi(\gamma(t, u_1)) \geq \Phi(u_1) > 0 \geq \Phi(u_2) \quad \text{if } t \in [0, t_\gamma]$$

and $\gamma(t, u_1) \in U$ hence $\|\gamma(t, u_1)\| \geq 2r$ if $t \in [t_\gamma, 1]$. Therefore $\gamma(t, u_1) \neq u_2$ for every $t \in [0, 1]$ and every $(u_1, u_2) \in \partial B_r V_0^n \times B_r W_0^n$. We define $F_t(u_1, u_2) := \gamma(t, u_1) - u_2$ for $t \in [0, 1]$, $(u_1, u_2) \in \Omega_n$. Then the above observation says that $F_t(u_1, u_2) \notin 0$ for every $t \in [0, 1]$, $(u_1, u_2) \in \partial B_r V_0^n \times B_r W_0^n$. Moreover the property (2.11) implies that $\gamma(t, u_1) \neq u_2$ for every $t \in [0, 1]$, $(u_1, u_2) \in B_r V_0^n \times \partial B_r W_0^n$. So we obtain

$$F_t(u_1, u_2) = \gamma(t, u_1) - u_2 \neq 0 \quad \text{for every } t \in [0, 1], (u_1, u_2) \in \partial \Omega_n.$$

Hence by the homotopy invariance of the degree,

$$(2.12) \quad 0 \neq \deg(P^1 - P^2, \Omega_n, 0) = \deg(F_0, \Omega_n, 0) = \deg(F_1, \Omega_n, 0),$$

where $P^1: E_n \rightarrow V_0^n$ and $P^2: E_n \rightarrow W_0^n$ are the orthogonal projections. On the other hand because of $F_1(u_1, u_2) = \gamma(1, u_1) - u_2$ and $\gamma(1, u_1) \in U$ for every $u_1 \in B_r V_0^n$, we have $F_1(u_1, u_2) \neq 0$ for every $(u_1, u_2) \in \Omega_n$. Hence $\deg(F_1, \Omega_n, 0) = 0$ by the property of the degree. So this is a contradiction.

In the case of (ii), we put $G_t(u_1, u_2) := \xi_n(t, u_2) - u_1$ and $F_t(u_1, u_2) := \xi_n(1, u_2) - \gamma(t, u_1)$ for $t \in [0, 1]$, $(u_1, u_2) \in \Omega_n$. If we suppose that γ satisfies (A2) for U and

$$\gamma([0, 1] \times B_r V_0^n) \cap \xi_n(1, \partial B_r W_0^n) = \emptyset,$$

then we can similarly prove that

$$0 \neq \deg(G_0, \Omega_n, 0) = \deg(G_1, \Omega_n, 0) = \deg(F_0, \Omega_n, 0) = \deg(F_1, \Omega_n, 0) = 0,$$

which is a contradiction to (2.12). ■

The following lemma which is necessary to prove our main result was stated in [12].

Lemma 13 ([12, Lemma 11]) *If Φ satisfies $(\Phi 4)$ with $\{E_n\}_n$ being compatible with respect to $(V_\infty \oplus W_\infty)$, then $\Phi|_{E_n}$ satisfies $(\Phi 4)$ with $(V_\infty \cap E_n, W_\infty \cap E_n)$ instead of $(V_\infty \oplus W_\infty)$ for every $n \in \mathbb{N}$.*

Now we state our main result.

Theorem 14 *Let Φ be a C^1 class functional on a Hilbert space E and let the conditions $(\Phi 1)$ to $(\Phi 4)$ be satisfied with $\{E_n\}_n$ in $(\Phi 1)$ compatible with the decomposition $V_0 \oplus W_0$ in $(\Phi 3)$ and $V_\infty \oplus W_\infty$ in $(\Phi 4)$ (cf.(2.10)). Moreover, suppose*

$$(2.13) \quad \limsup_{n \rightarrow \infty} |\dim E_n \cap V_\infty - \dim E_n \cap V_0| > 0.$$

holds. Then Φ has at least one non-zero critical point.

Remark 15 In [12], under the assumption (2.13) replaced by the assumption

$$(2.14) \quad \limsup_{n \rightarrow \infty} [\dim E_n \cap W_\infty - \dim E_n \cap W_0] > 0,$$

it is shown that Φ has at least one non-zero critical point. Therefore we shall only prove Theorem 14 under the assumption

$$\limsup_{n \rightarrow \infty} [\dim E_n \cap V_\infty - \dim E_n \cap V_0] > 0$$

by the compatibility of $\{E_n\}_n$ with the orthogonal decomposition $V_\infty \oplus W_\infty$.

Proof. By Remark 15, we may assume that

$$(2.15) \quad \limsup_{n \rightarrow \infty} [\dim E_n \cap V_\infty - \dim E_n \cap V_0] > 0.$$

We prove this theorem by contradiction. So suppose that there exist no critical points other than the origin. We fix $R_2 \geq \max\{R_1, 2r\}$ where R_1 is a constant satisfying (i), (ii) of $(\Phi 4)$. We define $U_1 := \{(v_\infty, w_\infty) \in V_\infty \oplus W_\infty; \|v_\infty\| < \max\{R_2, \delta\|w_\infty\|^\lambda\}\}$ and $U_2 := \{(v_\infty, w_\infty) \in V_\infty \oplus W_\infty; \|v_\infty\| > \max\{R_2, \delta\|w_\infty\|^\lambda\}\}$. Because of the assumption $(\Phi 2)$ and $(\Phi 4)$, $C_0 := \sup\{\Phi(u) | u \in \overline{U_1}\}$ is well defined.

First we consider the easier case where (i) of $(\Phi 3)$ holds. Then there exist an $r > 0$ and an $\varepsilon > 0$ satisfying (2.7). Suppose $\dim V_0 > 0$. Then $\dim E_n \cap V_0 > 0$ for large n because of the compatibility of $\{E_n\}_n$ with the orthogonal decomposition $V_0 \oplus W_0$. By the assumption (2.15), there exists an increasing sequence $\{n_j\}_j$ of natural numbers satisfying $\dim E_{n_j} \cap V_\infty - \dim E_{n_j} \cap V_0 > 0$. We may also assume that $\dim E_{n_j} \cap V_0 > 0$. By Lemma 8, there exist some $b_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$\|\nabla \Phi_n(u)\| \geq b_1 \quad \text{if } u \in \Phi_n^{-1}([-C_0 - 1, -\varepsilon/2]) \cup \Phi_n^{-1}([\varepsilon/2, C_0 + 1])$$

for every $n \geq n_1$. By Proposition 4 and Lemma 13, there exists some p.g.v.f. V_n for Φ_n satisfying the conditions (i), (ii) of $(\Phi 4)$ with $\nabla \Phi_n$ replaced by V_n and $U_2 \cap E_n$ is invariant with respect to V_n for every $n \in \mathbb{N}$.

We consider the following differential equation for large j

$$\begin{cases} \frac{d}{dt} \eta_j(t, u) = \frac{V_{n_j}(\eta_j(t, u))}{\|V_{n_j}(\eta_j(t, u))\|}, \\ \eta_j(0, u) = u \in \partial B_r V_0^{n_j}. \end{cases}$$

We note that $\eta_j(t, u)$ is well defined for every $t \in [0, \infty)$ by $\|\frac{d}{dt} \eta_j(t, u)\| = 1$. Now we choose a T with $T > 3C_0/b_1$ and set $\sigma_j(t, u) := \eta_j(Tt, u)$. Then we obtain $C_0 < \Phi_{n_j}(\eta_j(T, u))$ and $\|\sigma_j(t, u)\| \leq r + T =: R_3$ for every $(t, u) \in [0, 1] \times \partial B_r V_0^{n_j}$. Therefore $\sigma_j(1, u) \in U_2$ by the definition of C_0 .

We define for $u \in \partial B_r V_0^{n_j}$

$$\sigma_j(t, u) := \begin{cases} (t-1)P_\infty \sigma_j(1, u) + (2-t)\sigma_j(1, u) & \text{if } t \in [1, 2], \\ (3-t)P_\infty \sigma_j(1, u) + (t-2)R_3 \frac{P_\infty \sigma_j(1, u)}{\|P_\infty \sigma_j(1, u)\|} & \text{if } t \in [2, 3], \end{cases}$$

where P_∞ is the orthogonal projection onto V_∞ . We note that $\sigma_j(t, u) \in U_2$ for every $(t, u) \in [1, 3] \times \partial B_r V_0^{n_j}$ and $\|\sigma_j(t, u)\| \leq R_3$ for every $(t, u) \in [0, 3] \times \partial B_r V_0^{n_j}$.

Then $\sigma_j(3, \cdot)$ is a continuous map from $\partial B_r V_0^{n_j}$ to $\partial B_{R_3} V_\infty^{n_j}$. Therefore $\sigma_j(3, \cdot)$ is homotopic in $\partial B_r V_\infty^{n_j}$ to a constant map because of $\dim V_\infty^{n_j} > \dim V_0^{n_j}$. (cf.[6]) Denoting this homotopy by $H_j(t, u)$ for $(t, u) \in [0, 1] \times \partial B_r V_0^{n_j}$ with $H_j(0, u) = \sigma_j(3, u)$ and $H_j(1, u) = a_j$ for a fixed point $a_j \in \partial B_{R_3} V_\infty^{n_j}$. Now we define for $(t, u) \in \partial([0, 1] \times B_r V_0^{n_j})$

$$\gamma_j(t, u) := \begin{cases} u & \text{if } u \in B_r V_0^{n_j}, t = 0, \\ \sigma_j(4t, u) & \text{if } u \in \partial B_r V_0^{n_j}, t \in (0, 3/4], \\ H_j(4t - 3, u) & \text{if } u \in \partial B_r V_0^{n_j}, t \in (3/4, 1], \\ a_j & \text{if } u \in B_r V_0^{n_j}, t = 1. \end{cases}$$

Then γ_j is a continuous map from $\partial([0, 1] \times B_r V_0^{n_j})$ to E_{n_j} . Note that by the Dugundij extension theorem, there exists $\rho_j \in C([0, 1] \times B_r V_0^{n_j}, E_{n_j})$ such that

$$\begin{aligned} \rho_j(t, u) &= \gamma_j(t, u) \quad \text{if } (t, u) \in \partial([0, 1] \times B_r V_0^{n_j}), \\ \|\rho_j(t, u)\| &\leq R_3 \quad \text{for every } (t, u) \in [0, 1] \times B_r V_0^{n_j}. \end{aligned}$$

Because of the assumption $(\Phi 2)$, $C_1 := \inf\{\Phi(u) \mid \|u\| \leq R_3\}$ is well defined. Set

$$\Sigma_{n_j}^1 := \{\gamma \in C([0, 1] \times B_r V_0^{n_j}, E_{n_j}) \mid \gamma \text{ satisfies (A1) for } U_2\},$$

$$c_{n_j}^1 := \sup_{\gamma \in \Sigma_{n_j}^1} \min_{(t, u)} \Phi_{n_j}(\gamma(t, u)).$$

Since ρ_j constructed above belongs to $\Sigma_{n_j}^1$, $\Sigma_{n_j}^1 \neq \emptyset$ and $c_{n_j}^1 \geq C_1$. Using Lemma 12, we have $\gamma([0, 1] \times B_r V_0^{n_j}) \cap \partial B_r W_0^{n_j} \neq \emptyset$ for every $\gamma \in \Sigma_{n_j}^1$. Hence we obtain $\min_{B_r V_0} \Phi = 0 > -\varepsilon = \sup_{\partial B_r W_0^{n_j}} \Phi \geq c_{n_j}^1 \geq C_1$. Note that we may apply Lemma 3 with E, Φ, M, M_0, U replaced by $E_{n_j}, \Phi_{n_j}, B_r V_0^{n_j}, \partial B_r V_0^{n_j}, U_2 \cap E_{n_j}$ respectively, since $U_2 \cap E_{n_j}$ is an invariant set with respect to V_{n_j} . Hence for every j large enough, there exists some $v_j \in E_{n_j}$ such that

$$-\frac{\varepsilon}{2} \geq \Phi_{n_j}(v_j) \geq C_1 - 1 \quad \text{and} \quad \|\nabla \Phi_{n_j}(v_j)\| \leq \frac{1}{j}.$$

This contradicts Lemma 8.

In the case where $V_0 = \{0\}$, then $\Phi(0) = 0$ and $\Phi(u) \leq -\varepsilon$ for every u with $\|u\| = r$. Set

$$\Sigma_j := \{\gamma \in C([0, 1], E_{n_j}); \gamma(0) = 0, \gamma(1) \in U_2\} \neq \emptyset.$$

Then we have $\gamma([0, 1]) \cap \partial B_r E_{n_j} \neq \emptyset$ for every $\gamma \in \Sigma_j$. Hence we obtain

$$-\varepsilon \geq c_j := \sup_{\gamma \in \Sigma_j} \min_{t \in [0, 1]} \Phi(\gamma(t, u)) \geq \inf\{\Phi(u) \mid \|u\| \leq R_2 + 1\} > -\infty.$$

By applying Lemma 3 with $M = \{0\}$ and $M_0 = \emptyset$, we can show that for every j large enough, there exists some $v_j \in E_{n_j}$ such that

$$-\frac{\varepsilon}{2} \geq \Phi_{n_j}(v_j) \geq \inf\{\Phi(u) \mid \|u\| \leq R_2 + 1\} - 1 \text{ and } \|\nabla \Phi_{n_j}(v_j)\| \leq \frac{1}{j}$$

Therefore the same contradiction occurs.

The case where (ii) of $(\Phi 3)$ holds could be treated similarly by using a deformation. Assume that 0 is the only critical point, then we let ξ_{n_j} and η_{n_j} be mappings satisfying (i) \sim (vii) in Lemma 10. Set

$$\Sigma_{n_j}^2 := \{\gamma \in C([0, 1] \times B_r V_0^{n_j}) \mid \gamma \text{ satisfies (A2) for } U_2\}.$$

Then we can similarly obtain $\Sigma_{n_j}^2 \neq \emptyset$ and the proof similarly goes on by using Lemma 12 and Lemma 3. ■

2.3. The existence of critical points for even functional

Now we consider the case where Φ is even. At first, we recall the definition of the Krasnoselskii genus.

Definition 16 ([15, 5.1 Definition]) *Let E be a Hilbert space and set*

$$\Sigma := \{A \subset E; A \text{ is closed, } A = -A\}.$$

We define for $A \in \Sigma$

$$i(A) := \begin{cases} \inf \{m \mid \text{there exists an odd mapping } h \in C(A, \mathbb{R}^m \setminus \{0\})\} \\ 0, & \text{if } A = \emptyset \\ \infty, & \text{otherwise} \end{cases}$$

It is well known that the genus has the following properties.

Proposition 17 ([15, 5.4 Proposition]) *Let $A, B \in \Sigma$ and $h \in C(E, E)$ be an odd map. Then the following hold:*

- (i) $i(A) \leq i(B)$ if $A \subset B$.
- (ii) $i(A \cup B) \leq i(A) + i(B)$.
- (iii) $i(A) \leq i(\overline{h(A)})$.
- (iv) *If A is compact, there exists a neighborhood $N \in \Sigma$ such that $A \subset N^\circ \subset N$ and $i(N) = i(A)$.*

- (v) If F is a linear subspace of E with $\dim F = n$, and if $A \subset F$ is bounded, open and symmetric neighborhood of the origin in F , then $i(\partial A) = n$.

We also recall the usual Palais-Smale condition.

Definition 18 Let X be a Hilbert space and let $\Phi : X \rightarrow \mathbb{R}$ be a C^1 class functional. We say that Φ satisfies $(PS)_c$ condition for $c \in \mathbb{R}$ if every $\{u_n\} \subset X$ satisfying $\Phi(u_n) \rightarrow c$ and $\nabla \Phi(u_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence.

Now we state an assumption to describe our result when Φ is even:

($\Phi 0$) $\Phi_n \in C^1(E_n, \mathbb{R})$ satisfies $(PS)_c$ condition for every $n \in \mathbb{N}$ and $c \in \mathbb{R}$.

We introduce the following notations:

$$\begin{aligned} K &:= \{u \in E \mid \nabla \Phi(u) = 0\}, & K_c &:= \{u \in K \mid \Phi(u) = c\} \\ K([a, b]) &:= \{u \in K \mid \Phi(u) \in [a, b]\}. \end{aligned}$$

Theorem 19 Let $\Phi \in C^1(E, \mathbb{R})$ be even and let ($\Phi 0$), ($\Phi 1$) and ($\Phi 3$) hold. Moreover, suppose that there exists a subspace V_∞ with $\inf_{V_\infty} \Phi > -\infty$ and

$$(2.16) \quad k := \limsup_{n \rightarrow \infty} [\dim E_n \cap W_0 - \dim E_n \cap W_\infty] > 0,$$

where $W_\infty := V_\infty^\perp$. Also assume that $\{E_n\}_n$ in ($\Phi 1$) is compatible with the orthogonal decompositions $V_\infty \oplus W_\infty$ and $V_0 \oplus W_0$ (cf.(2.10)). Then Φ has at least k pairs of non-trivial critical points in $\Phi^{-1}((-\infty, 0])$.

Proof. We may assume that Φ has only finitely many critical points in $\Phi^{-1}((-\infty, 0])$. Because of the assumption (2.16), there exists an increasing sequence $\{n_j\}_j \subset \mathbb{N}$ such that $\dim(E_{n_j} \cap W_0) = k + \dim(E_{n_j} \cap W_\infty)$.

We define for j sufficiently large and $1 \leq l \leq k$

$$\Sigma_j^l := \{A \subset E_{n_j}; \text{ compact, } A = -A, \infty > i(A) \geq \dim(E_{n_j} \cap W_\infty) + l\},$$

$$c_j^l := \inf_{A \in \Sigma_j^l} \max_{u \in A} \Phi(u).$$

We set $-M := \inf_{V_\infty} \Phi$. Then we can prove $M > 0$, that is $0 > \inf_{V_\infty} \Phi$. Indeed, suppose that this is not the case, then $M = 0$. Note $\inf_{V_\infty} \Phi = 0$. If the case of (i) in ($\Phi 3$) holds, we assume that $\varepsilon > 0$ is a constant satisfying (2.7). Then from $i(\partial B_r W_0^{n_j}) \geq \dim(W_\infty \cap E_{n_j}) + k$ and the compatibility of $\{E_n\}_n$, we have $\text{TokyoJ.Math(toappear).} A \cap (V_\infty \cap E_{n_j}) \neq \emptyset$ for every $A \in \Sigma_j^1$. Therefore we obtain $0 \leq c_j^1 \leq \dots \leq c_j^k \leq -\varepsilon < 0$. This is a contradiction. Next if the case (ii) of ($\Phi 3$) holds, since Φ has only finitely many critical

points in $\Phi^{-1}((-\infty, 0])$, then there exist an $\varepsilon_0 > 0$ and some $r_0 > 0$ such that $K([-\varepsilon_0, 0]) \cap B_{2r_0}E = \{0\}$ and $r \geq r_0$, where $r > 0$ is a constant satisfying (2.6). Then by $(\Phi 1)$ and (ii) of $(\Phi 3)$, there exist $n_0 \in \mathbb{N}$ and $d > 0$ such that

$$\|\nabla \Phi_n(u)\| \geq d \quad \text{for } \forall u \in \Phi_n^{-1}([-\varepsilon_0, 0]) \cap B_{2r_0}E \setminus B_{r_0/2}E$$

for every $n \geq n_0$. Therefore by the standard deformation argument, there exist $(\varepsilon_0 \geq) \varepsilon > 0$ and $\eta_n \in C([0, 1] \times E_n, E_n)$ for every $n \geq n_0$ satisfying (a) \sim (e); (a) $\eta_n(0, u) = u$ for every $u \in E_n$, (b) $\Phi_n(\eta_n(1, u)) \leq -\varepsilon$ for every $u \in \partial B_{r_0}W_0^n$ and (c) $\eta_n(t, u)$ is odd in u for every $t \in [0, 1]$. Moreover, we have $i(\eta_n(1, \partial B_{r_0}W_0^n)) \geq i(\partial B_{r_0}W_0^n) = \dim W_0^n$ by using (iii) and (v) in Proposition 17. This yields the same contradiction as in the case (i) of $(\Phi 3)$. Hence we obtain $M > 0$, that is $\inf_{V_\infty} \Phi < 0$.

Because $\{E_n\}_n$ is compatible with the decomposition $V_\infty \oplus W_\infty$, c_j^l is well defined for $1 \leq l \leq k$ and

$$(2.17) \quad c_j^l \in [-M, -\varepsilon] \quad (1 \leq l \leq k),$$

where $\varepsilon > 0$ is some constant independ of j from above argument and $(\Phi 3)$. By the standard argument (cf. Theorem 6.1 in [13]), it is shown that $c_j^l \in [-M, -\varepsilon]$ ($1 \leq l \leq k$) are critical values of Φ_{n_j} , since Φ_{n_j} satisfies $(PS)_c$ for every $c \in \mathbb{R}$. Then, by taking a subsequence if necessary, we may assume that there exist $c^l \in [-M, -\varepsilon]$ ($1 \leq l \leq k$) such that $c_j^l \rightarrow c^l$ as $(j \rightarrow \infty)$ for $1 \leq l \leq k$. We note $c^1 \leq c^2 \leq \dots \leq c^k$. Then we have $K_{c^l} \neq \emptyset$ for $1 \leq l \leq k$ since $(\Phi 1)$ holds and c_j^l is a critical values of Φ_{n_j} . Suppose that $1 \leq l \leq k$ and $m(l) =: m \in \mathbb{N} \cup \{0\}$ the largest integer such that $l + m \leq k$ and $c := c^l = \dots = c^{l+m}$. There exists some $\delta_0 > 0$ such that

$$K([c - \delta_0, c + \delta_0]) \setminus K_c = \emptyset$$

since c is an isolated critical value. Let $K_c := \{\pm u_1, \dots, \pm u_p\}$ with $u_i \neq u_j$ for every $i \neq j$. Next we choose weakly open convex neighborhoods N_q of u_q ($1 \leq q \leq p$) such that

$$(2.18) \quad \text{dist}(\overline{N_q}, -\overline{N_q}) > 0 \text{ and } \text{dist}(\overline{N_{q'}}, \overline{N_q} \cup -\overline{N_q}) > 0 \quad (q \neq q').$$

Indeed, we shall show (2.18) in l^2 because E is a separable Hilbert space. So let $u_i := (u_n^i)_{n=1}^\infty \in l^2$ ($1 \leq i \leq p$). Since $u_i \neq u_j$ for every $i \neq j$, there exists some $N \in \mathbb{N}$ such that $u_i^N \neq u_j^N$ in \mathbb{R}^N ($i \neq j$), where $u_i^N := (u_n^i)_{n=1}^N$ ($1 \leq i \leq p$). Therefore there exists some $\delta > 0$ such that

$$\text{dist}(\overline{B_q}, -\overline{B_q}) > 0 \text{ and } \text{dist}(\overline{B_{q'}}, \overline{B_q} \cup -\overline{B_q}) > 0 \quad (q \neq q'),$$

where $B_q := B_\delta(u_q^N) := \{v \in \mathbb{R}^N \mid \text{dist}(u_q^N, v) < \delta\}$. Hence we can choose $N_q := \{v \in l^2 \mid \text{dist}(u_q^N, v^N) < \delta\}$.

Next we choose $\delta_1 > 0$ satisfying

$$\text{dist}((N_q)_{\delta_1}, (\pm N_{q'})_{\delta_1}) > 0 \ (q \neq q') \text{ and } \text{dist}((N_q)_{\delta_1}, (-N_q)_{\delta_1}) > 0,$$

where $\pm N_q := N_q \cup (-N_q)$ and $(N_q)_{\delta_1} := \{u \in E \mid \text{dist}(u, N_q) \leq \delta_1\}$. We set $N := (\pm N_1) \cup \cdots \cup (\pm N_p)$. Then we can easily see that $i((N)_{\delta_1}) = 1$. On the other hand, the assumption $(\Phi 1)$, there exists some $b > 0$ and $n_1 \in \mathbb{N}$ such that

$$\|\nabla \Phi_n(u)\| \geq b \quad \text{if } u \in \Phi_n^{-1}([c - \delta_0, c + \delta_0]) \setminus N$$

for every $n \geq n_1$. Indeed, we assume that for every $n \in \mathbb{N}$ there exists some $u_n \in \Phi_n^{-1}([c - \delta_0, c + \delta_0]) \setminus N$ such that $\|\nabla \Phi_n(u_n)\| \leq 1/n$. Then we have some subsequence $\{u_{n_j}\}$ of $\{u_n\}$ being a $(WPS)_{c'}^*$ sequence ($c' \in [c - \delta_0, c + \delta_0]$). By the assumption $(\Phi 1)$, $\{u_{n_j}\}$ has a subsequence weakly convergent to a critical point u_0 of Φ with $\Phi(u_0) = c'$. On the other hand, since N is a weakly open neighborhood of K_c , we obtain $u_0 \notin N$, that is $u_0 \notin K_c$ and $c \neq c'$. This is a contradiction because c is the unique critical value in $[c - \delta_0, c + \delta_0]$.

Then we put $\varepsilon_1 := \min\{\delta_0/3, \delta_1 b/12, b/6\}$, by the standard deformation argument, for every $n \geq n_1$, there exists $\eta_n \in C([0, 1] \times E_n, E_n)$ satisfying (a) \sim (e):

- (a) $\eta_n(t, u)$ is non-increasing in t for every $u \in E_n$,
- (b) $\eta_n(t, u) = u$ for $t \in [0, 1]$, $u \notin \Phi_n^{-1}([c - 3\varepsilon_1, c + 3\varepsilon_1])$ and also for $t \in [0, 1]$, $u \in N$,
- (c) $\eta_n(1, \overline{D \setminus (N)_{\delta_1}}) \subset \Phi^{c - \varepsilon_1}$ if $D \subset E_n$ satisfies $D \subset \Phi_n^{c + \varepsilon_1}$,
- (d) $\eta_n(t, u)$ is odd in u for every $t \in [0, 1]$,
- (e) $\eta_n(t, \cdot)$ is a homeomorphism from E_n to E_n for every $t \in [0, 1]$

Because c_j^l and c_j^{l+m} are convergent to c , there exists some $n_2 \geq n_1$ such that $c_j^l, c_j^{l+m} \in [c - \varepsilon_1/2, c + \varepsilon_1/2]$ for every j satisfying $n_j \geq n_2$. Now we fix j such that $n_j \geq n_2$. By the definition of c_j^{l+m} , there exists some $D \in \Sigma_j^{l+m}$ such that $\max_D \Phi_{n_j} \leq c + \varepsilon_1$. Using above (c), we obtain $\eta_{n_j}(1, \overline{D \setminus (N)_{\delta_1}}) \subset \Phi^{c - \varepsilon_1}$. Hence

$$i\left(\eta_{n_j}(1, \overline{D \setminus (N)_{\delta_1}})\right) \leq \dim(E_{n_j} \cap W_\infty) + (l - 1)$$

holds, by the definition of c_j^l and $c - \varepsilon_1 < c_j^l$. Therefore we obtain

$$\begin{aligned} \dim(E_{n_j} \cap W_\infty) + l + m &\leq i(D) \leq i(\overline{D \setminus (N)_{\delta_1}}) + i((N)_{\delta_1}) \\ &\leq i\left(\eta_{n_j}(1, \overline{D \setminus (N)_{\delta_1}})\right) + 1 \\ &\leq \dim(E_{n_j} \cap W_\infty) + (l - 1) + 1. \end{aligned}$$

Hence we have $m = 0$. This yields $-M \leq c^1 < c^2 < \cdots < c^k \leq -\varepsilon$. Therefore Φ has at least k pairs of non-trivial critical points in $\Phi^{-1}((-\infty, 0])$. \blacksquare

§3. Applications

Let us return to the nonlinear wave equation (WE):

$$(WE) \quad \begin{cases} \square u(x, t) = h(x, t, u(x, t)), & (0 < x < \pi, t \in \mathbb{R}), \\ u(0, t) = u(\pi, t) = 0, & (t \in \mathbb{R}), \\ u(x, t + 2\pi) = u(x, t), & (0 < x < \pi, t \in \mathbb{R}). \end{cases}$$

The nonlinear term $h: [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is assumed to satisfy the following conditions (h1) to (h3).

(h1) h is continuous and $h(x, t + 2\pi, \xi) = h(x, t, \xi)$ $((x, t, \xi) \in [0, \pi] \times \mathbb{R}^2)$.

(h2) h is non-decreasing in ξ and $h(x, t, \xi) \neq 0$ ($\xi \neq 0$).

(h3) There exist constants $b_0 \geq 0$, $b > 0$ that satisfy the following properties:

$$\begin{aligned} g_0(x, t, \xi) &:= h(x, t, \xi) - b_0 \xi = o(|\xi|) \text{ as } \xi \rightarrow 0 \text{ uniformly in } (x, t), \\ g(x, t, \xi) &:= h(x, t, \xi) - b \xi = o(|\xi|) \text{ as } |\xi| \rightarrow \infty \text{ uniformly in } (x, t). \end{aligned}$$

Let $Q := (0, \pi) \times (0, 2\pi)$ and define

$$b_0^+ := \min \{ \lambda \mid \lambda \in \sigma(\square), b_0 < \lambda \}, \quad b_0^- := \max \{ \lambda \mid \lambda \in \sigma(\square), \lambda < b_0 \},$$

where \square (D'Alembertian) means the self-adjoint operator in $L^2(Q)$ obtained as the closure of $\partial^2/\partial t^2 - \partial^2/\partial x^2$ with domain $\{ u \in C^2([0, \pi] \times \mathbb{R}) \mid u(x, t + 2\pi) = u(x, t), u(0, t) = u(\pi, t) = 0 \}$.

Theorem 20 *Assume that the non-linear term h of the equation (WE) satisfies the conditions (h1) \sim (h3) and let b_0 , g_0 , b and g be as in (h3). Set $G(x, t, \xi) := \int_0^\xi g(x, t, s) ds$, $G_0(x, t, \xi) := \int_0^\xi g_0(x, t, s) ds$ and consider the following conditions:*

(C1) g is bounded, and $G(x, t, \xi) \rightarrow +\infty$ (as $|\xi| \rightarrow \infty$) uniformly in (x, t) ,

(C2) the following condition (a1) or (a2) holds for some constants $0 < \alpha \leq \beta < 1$ satisfying $\beta - \frac{\alpha}{2} < \frac{1}{2}$, $c_1, c_2 > 0$, and $d_1, d_2 \geq 0$:

$$\begin{aligned} (a1) \quad & |g(x, t, \xi)| \leq c_1 |\xi|^\beta + d_1, \quad G(x, t, \xi) \geq c_2 |\xi|^{\alpha+1} - d_2 |\xi|, \\ (a2) \quad & |g(x, t, \xi)| \leq c_1 |\xi|^\beta + d_1, \quad G(x, t, \xi) \leq -c_2 |\xi|^{\alpha+1} + d_2 |\xi|. \end{aligned}$$

(C3) There exists a $\delta > 0$ such that $G_0(x, t, \xi) \geq 0$ if $|\xi| \leq \delta$,

(C4) There exists a $\delta > 0$ such that $G_0(x, t, \xi) \leq 0$ if $|\xi| \leq \delta$.

Then (WE) has a non-trivial weak solution in each of the following cases (A1) to (A4):

- (A1) $b_0 \notin \sigma(\square)$, $b \notin \sigma(\square)$ and $b \notin [b_0^-, b_0^+)$;
- (A2) $b_0 \in \sigma(\square)$, $b \notin \sigma(\square)$, and one of the following conditions hold:
- (1) $b \notin [b_0, b_0^+)$ and (C3);
 - (2) $b \notin [b_0^-, b_0)$ and (C4);
- (A3) $b_0 \notin \sigma(\square)$, $b \in \sigma(\square)$, and one of the following conditions holds:
- (1) $b \notin [b_0^-, b_0]$ and (C1) or (a1) of (C2);
 - (2) $b \notin [b_0, b_0^+]$ and (a2) of (C2);
- (A4) $b_0 \in \sigma(\square)$, $b \in \sigma(\square)$, and one of the following conditions holds:
- (1) (C3), $b_0 \neq b$ and (C1) or (a1) of (C2);
 - (2) (C3), $b_0^+ \neq b$ and (a2) of (C2);
 - (3) (C4), $b_0^- \neq b$ and (C1) or (a1) of (C2);
 - (4) (C4), $b_0 \neq b$ and (a2) of (C2);

Remark 21 In the nonresonant case ($b \notin \sigma(\square)$), Theorem 20 is contained in the results of [2] and [12].

In the resonant case ($b \in \sigma(\square)$), the condition imposed on b_0 and b is more general than the result of [12]. In [12], provided the case (1) of (A4), the existence of a weak nontrivial solution to (WE) is proved only under the condition of $b_0 < b$. On the other hand, Theorem 20 shows that the condition $b_0 > b$ in the case (1) of (A4) also implies the existence of a nontrivial solution.

We note that $b_0 = 0$ yields condition (C3) by the assumption (h2) and $h(x, t, \xi) = g_0(x, t, \xi)$.

Although the difficult part of the proof of Theorem 20 relies on Theorem 14 in Section 2, the fundamental plan of proof is almost parallel to that in [12]. So, we only give the proof for the case of assumption (1) of (A3). First we recall the variational setting for the existence of a weak solution to (WE).

By the Fourier series expansion, every real-valued $u \in L^2(Q)$ can be written as

$$u(x, t) = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} u_{kj} \sin jx e^{ikt}$$

with $\overline{u_{kj}} = u_{-kj}$ for all j, k . Using this expansion, we set

$$\|u\|_E := \left(\pi^2 \sum_{j \neq |k|} |j^2 - k^2| |u_{kj}|^2 + \pi^2 \sum_{j=|k|} |u_{kj}|^2 \right)^{1/2}$$

and we define the space E by

$$E := \{ u \in L^2(Q) \mid \|u\|_E < \infty \}.$$

Then E is a Hilbert space with the inner-product

$$\langle u, v \rangle_E := \pi^2 \sum_{j \neq |k|} |j^2 - k^2| u_{kj} \overline{v_{kj}} + \pi^2 \sum_{j=|k|} u_{kj} \overline{v_{kj}},$$

where u_{kj}, v_{kj} are Fourier coefficients of u, v respectively. E has an orthogonal decomposition $E = E^+ \oplus E^0 \oplus E^-$ where

$$E^+ := \left\{ u \in E : u(x, t) = \sum_{j^2 > k^2} u_{kj} \sin jx e^{ikt} \right\},$$

$$E^0 := \left\{ u \in E : u(x, t) = \sum_{j^2 = k^2} u_{kj} \sin jx e^{ikt} \right\},$$

and

$$E^- := \left\{ u \in E : u(x, t) = \sum_{j^2 < k^2} u_{kj} \sin jx e^{ikt} \right\}.$$

The orthogonal projections onto E^- , E^0 and E^+ are designated by P^- , P^0 and P^+ , respectively.

It is well known that the inclusions $E^\pm \hookrightarrow L^2(Q)$ are compact and E^0 is a closed subspace of $L^2(Q)$.

For each $n \in \mathbb{N}$, we set

$$E_n := \text{span} \{ \sin jx \sin kt, \sin jx \cos kt : 0 < j \leq n, |k| \leq n \}.$$

Then $\{E_n\}_n$ is an increasing sequence of finite dimensional subspace of E with $\bigcup_{n=1}^\infty E_n$ being dense in E . Let us note that this sequence is compatible with the decomposition $E = E^- \oplus E^0 \oplus E^+$, i.e., the orthogonal projection onto E_n commutes with P^- , P^0 and P^+ for every n .

Consider the functional Φ defined on E by

$$(3.1) \quad \Phi(u) := \frac{1}{2} \int_Q (u_x^2 - u_t^2) dx dt - \int_Q H(x, t, u) dx dt$$

$$(3.2) \quad = \frac{1}{2} (\|P^+ u\|^2 - \|P^- u\|^2) - \Psi(u),$$

where $H(x, t, \xi) := \int_0^\xi h(x, t, s) ds$, $\Psi(u) := \int_Q H(x, t, u) dx dt$. Under the conditions (h1) to (h3), it is clear that $\Phi(u)$ is a C^1 class functional on E with $\langle \nabla \Phi(u), v \rangle_E = \langle (P^+ - P^-)u, v \rangle_E - \int_Q h(x, t, u(x, t))v(x, t) dx dt$.

It is well known that a critical point of Φ is a weak solution to (WE).

The proof for the case (1) of (A3)

We shall show that the functional Φ defined by (3.2) satisfies the assumptions $(\Phi 1) \sim (\Phi 4)$ in Theorem 14 and the dimension condition (2.14) or (2.15) in Remark 15 holds.

In [2], it is shown that Φ satisfies condition (ii) of $(\Phi 3)$ with respect to $(V_0, W_0) = (X_0^+, X_0^-)$, where

$$X_0^+ := \left\{ w \in E : w(x, t) = \sum_{j^2 - k^2 > b_0} u_{kj} \sin jx e^{ikt} \right\},$$

$$X_0^- := \left\{ w \in E : w(x, t) = \sum_{j^2 - k^2 < b_0} u_{kj} \sin jx e^{ikt} \right\}.$$

Moreover, in [12], it is shown that Φ satisfies $(\Phi 1)$, $(\Phi 2)$ and $(\Phi 4)$ with respect to $(V_\infty, W_\infty) = (X^+, X^- \oplus X^0)$, where

$$X^+ := \left\{ u \in E : u(x, t) = \sum_{j^2 - k^2 > b} u_{kj} \sin jx e^{ikt} \right\},$$

$$X^0 := \left\{ u \in E : u(x, t) = \sum_{j^2 - k^2 = b} u_{kj} \sin jx e^{ikt} \right\},$$

$$X^- := \left\{ u \in E : u(x, t) = \sum_{j^2 - k^2 < b} u_{kj} \sin jx e^{ikt} \right\}.$$

Finally we check the dimension condition. First we suppose that $b < b_0^-$. Let $E(\lambda)$ be the eigenspace of $\lambda \in \sigma(\square)$. Then the definition of X^+ , X_0^+ and the assumption $b < b_0^-$ imply $X_0^+ \oplus E(b_0^-) \subset X^+$. Moreover $E(b_0^-) \subset E_n$ for large $n \in \mathbb{N}$. Therefore, if n is large enough, we obtain

$$E(b_0^-) \oplus (E_n \cap X_0^+) = E_n \cap (X_0^+ \oplus E(b_0^-)) \subset E_n \cap X^+.$$

Hence

$$\liminf_{n \rightarrow \infty} [\dim(E_n \cap V_\infty) - \dim(E_n \cap V_0)] \geq \dim E(b_0^-) > 0.$$

Next we assume $b > b_0$. Then we similarly have $X_0^- \subset X^0 \oplus X^-$ and $E(b) \subset E_n$ for large $n \in \mathbb{N}$. Therefore, if n is large enough, we obtain

$$(E_n \cap X_0^-) \oplus E(b) = E_n \cap (X_0^- \oplus X^0) \subset E_n \cap (X^- \oplus X^0).$$

This implies

$$\liminf_{n \rightarrow \infty} [\dim(E_n \cap W_\infty) - \dim(E_n \cap W_0)] \geq \dim E(b) > 0.$$

■

If the nonlinear term h is odd in ξ , then we obtain the following result by applying Theorem 19 in Section 2. We omit the proof here because it is easy to check the assumptions in Theorem 19 for Φ defined by (3.1). Indeed, we can prove the boundness of Palais–Smale sequences of Φ_n defined on each finite dimension subspace E_n by the same argument as in [2, Proposition 2.6] and [12, Proposition 18].

Theorem 22 *Assume that the non-linear term h of the equation (WE) satisfies the conditions (h1) \sim (h3) and let b_0 , g_0 , b and g be as in (h3). Moreover we suppose that $h(x, t, \xi)$ is odd in ξ . Then (WE) has at least k pairs of weak solutions in each of the following cases (A1) to (A4):*

- (A1) $b_0 \notin \sigma(\square)$, $b \notin \sigma(\square)$ and $b \notin [b_0^-, b_0^+)$ with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_5$ (if $b \geq b_0^+$);
- (A2) $b_0 \in \sigma(\square)$, $b \notin \sigma(\square)$, and one of the following conditions hold:
 - (1) $b \notin [b_0, b_0^+)$ and (C3) with $k = \sharp K_3$ (if $b < b_0$), $k = \sharp K_5$ (if $b_0^+ \leq b$);
 - (2) $b \notin [b_0^-, b_0)$ and (C4) with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_6$ (if $b_0 \leq b$);
- (A3) $b_0 \notin \sigma(\square)$, $b \in \sigma(\square)$, and one of the following conditions holds:
 - (1) $b \notin [b_0^-, b_0]$ and (C1) or (a1) of (C2) with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_7$ (if $b_0 < b$);
 - (2) $b \notin [b_0, b_0^+]$ and (a2) of (C2) with $k = \sharp K_2$ (if $b < b_0$), $k = \sharp K_5$ (if $b_0^+ < b$);
- (A4) $b_0 \in \sigma(\square)$, $b \in \sigma(\square)$, and one of the following conditions holds:
 - (1) (C3), $b_0 \neq b$ and (C1) or (a1) of (C2) with $k = \sharp K_7$ (if $b_0 < b$), $k = \sharp K_3$ (if $b < b_0$);
 - (2) (C3), $b_0^+ \neq b$ and (a2) of (C2) with $k = \sharp K_4$ (if $b < b_0^+$), $k = \sharp K_5$ (if $b_0^+ < b$);
 - (3) (C4), $b_0^- \neq b$ and (C1) or (a1) of (C2) with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_8$ (if $b_0^- < b$);

- (4) (C4), $b_0 \neq b$ and (a2) of (C2)
 with $k = \sharp K_2$ (if $b < b_0$), $k = \sharp K_6$ (if $b_0 < b$);

where

$$\begin{aligned} K_1 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b < j^2 - k^2 < b_0\}, \\ K_2 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b \leq j^2 - k^2 < b_0\}, \\ K_3 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b < j^2 - k^2 \leq b_0\}, \\ K_4 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b \leq j^2 - k^2 \leq b_0\}, \\ K_5 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b_0 < j^2 - k^2 < b\}, \\ K_6 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b_0 \leq j^2 - k^2 < b\}, \\ K_7 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b_0 < j^2 - k^2 \leq b\}, \\ K_8 &:= \{(j, k) \in \mathbb{N} \times \mathbb{Z} \mid b_0 \leq j^2 - k^2 \leq b\}. \end{aligned}$$

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